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Determination of temperature field created by planar heat source in a solid body consisting of three parts in mutual thermal contact

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Abstract

We present a solution of, a one-dimensional heat conduction equation for a solid body consisting of three parts. The exterior parts possess identical thermal properties. The thermal properties of the inner part differ from the outer ones. The temperature field is created by a planar heat source located between one outer and one inner part. This solution can be used in measurements of thermal properties of an inner body by the pulse transient method assuming that the thermal properties of outer bodies are known.

Keywords: Thermal properties; Solid body; Planar heat source; Temperature field

1. Introduction

The two exterior parts (denoted as index 1 or 3, respectively) of the equal thermo-physical properties $\lambda_1 = \lambda_3$, $c_1 = c_3$, $\rho_1 = \rho_3$ (thermal conductivity, specific heat, and density) are considered as semi-infinite in intervals of (1), $x \in (-\infty, 0\rangle$, (3), $x \in \langle h, +\infty)$. The inner part (2) with thermal parameters λ_2, c_2, ρ_2 and thickness h is placed inside the interval $x \in \langle 0, h\rangle$. The planar heat source acts on the plane at $x = 0$ where parts 1 and 2 are in thermal contact (Fig. 1). The time dependence of the temperature is measured at the second interface at $x = h$, which is the plane of the thermal contact of parts 2 and 3.

This "three-body problem" has been solved before, as shown in Ref. [1]. The purpose of the current paper is to include detailed mathematical steps, which might be omitted in Ref. [1] as well as to sketch possible spreading of current theory to further heating methods.

2. Determination of the temperature field

In order to determine the temperature, one-dimensional heat conduction equations have to be solved

$$\frac{\partial T_1}{\partial t} - a_1 \frac{\partial^2 T_1}{\partial x^2} = 0, \quad \text{at an interval of } x \in (-\infty, 0) \tag{2.1}$$

$$\frac{\partial T_2}{\partial t} - a_2 \frac{\partial^2 T_2}{\partial x^2} = 0, \quad \text{at an interval of } x \in (0, h\rangle \tag{2.2}$$

$$\frac{\partial T_3}{\partial t} - a_1 \frac{\partial^2 T_3}{\partial x^2} = 0, \quad \text{at an interval of } x \in \langle h, +\infty) \tag{2.3}$$

under the following initial and boundary conditions

$$T_1(x, t=0) = T_2(x, t=0) = T_3(x, t=0) = 0 \tag{2.4}$$

$$T_1(x \to -\infty, t) = T_3(x \to +\infty, t) = 0 \tag{2.5}$$

$$T_1(x=0, t) - T_2(x=0, t) = 0 \tag{2.6}$$

$$T_2(x=h, t) - T_3(x=h, t) = 0 \tag{2.7}$$

$$\lambda_1 \frac{\partial T_1(x \to -0, t)}{\partial x} - \lambda_2 \frac{\partial T_2(x \to +0, t)}{\partial x} = q(t) \tag{2.8}$$

$$\lambda_2 \frac{\partial T_2(x=h, t)}{\partial x} - \lambda_1 \frac{\partial T_3(x=h, t)}{\partial x} = 0 \tag{2.9}$$

where $a_j = \lambda_j / c_j \rho_j$, is the diffusivity $j = 1, 2$, and $q(t)$ denotes the heat produced by the unit surface per unit time by the planar heat source at $x = 0$. In general, it is a given function of the time.

The Laplace transformation will be applied to Eqs. (2.1)–(2.9) to solve this problem. If we denote

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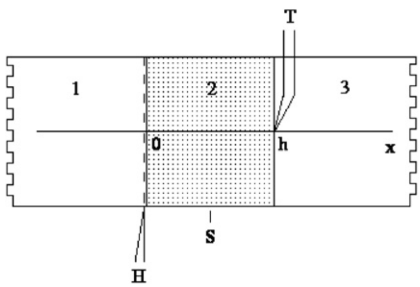

Fig. 1. A scheme of specimen S set up, H—heat source, T—thermometer, and $\boldsymbol{x}$—axis.

$$L\big[T_j(x,t)\big] = \int_0^\infty T_j(x,t)\exp(-st)\,dt = \vartheta_j(x,s)$$
$$j = 1,2,3 \tag{2.10}$$

and

$$L\big[q(t)\big] = p(s) \tag{2.11}$$

we obtain three ordinary differential equations

$$s\vartheta_1 - a_1\frac{\partial^2\vartheta_1}{\partial x^2} = 0, \qquad s\vartheta_2 - a_2\frac{\partial^2\vartheta_2}{\partial x^2} = 0$$
$$s\vartheta_3 - a_1\frac{\partial^2\vartheta_3}{\partial x^2} = 0 \tag{2.12}$$

where the initial condition (2.4) was accounted for. The boundary conditions are transformed as follows

$$\vartheta_1(x \to -\infty, s) = \vartheta_3(x \to +\infty, s) = 0 \tag{2.5a}$$

$$\vartheta_1(x = 0, s) - \vartheta_2(x = 0, s) = 0 \tag{2.6a}$$

$$\vartheta_2(x = h, s) - \vartheta_3(x = h, s) = 0 \tag{2.7a}$$

$$\lambda_1\frac{\partial\vartheta_1(x \to -0, s)}{\partial x} - \lambda_2\frac{\partial\vartheta_2(x \to +0, s)}{\partial x} = p(s) \tag{2.8a}$$

$$\lambda_2\frac{\partial\vartheta_2(x = h, s)}{\partial x} - \lambda_1\frac{\partial\vartheta_3(x = h, s)}{\partial x} = 0 \tag{2.9a}$$

The general solution of the transformed ordinary differential equation is

$$\vartheta_j(x,s) = A_j\exp\big(\sqrt{s/a_j}\,x\big) + B_j\exp\big(-\sqrt{s/a_j}\,x\big) \tag{2.13}$$

Here $j = 1,2,3$ and $a_1 = a_3$.

The six constants $A_1, B_1, A_2, B_2, A_3, B_3$ have to obey the transformed initial and boundary conditions.

(a) Additionally, we restrict ourselves to a planar heat source creating a constant heat flux q at a time interval $(0,\infty)$, then

$$q(t) = q\theta(t), \qquad q = const \tag{2.14}$$

where $\theta(\tau)$ is Heaviside's step-wise function ($\theta(t) = 0$ if $t < 0$ and $\theta(t) = 1$ for $t > 0$). The Laplace transformation of this function is

$$p(s) = q/s \tag{2.15}$$

In this special case the solution of the problem leads to a temperature field given by the formulae (see Appendices A and B)

(1) $x \in (-\infty, 0\rangle$,

$$T_{S1}(x,t) = \frac{2q}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\theta(t)\sqrt{t}\left[\mathrm{i}\Phi_c\left(\frac{-x + 2nh\sqrt{a_1/a_2}}{2\sqrt{a_1 t}}\right) - \frac{k_-}{k_+}\mathrm{i}\Phi_c\left(\frac{-x + 2(n+1)h\sqrt{a_1/a_2}}{2\sqrt{a_1 t}}\right)\right] \tag{2.16}$$

(2) $x \in \langle 0, h\rangle$,

$$T_{S2}(x,t) = \frac{2q}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\theta(t)\sqrt{t}\left[\mathrm{i}\Phi_c\left(\frac{2nh + x}{2\sqrt{a_2 t}}\right) - \frac{k_-}{k_+}\mathrm{i}\Phi_c\left(\frac{2(n+1)h - x}{2\sqrt{a_2 t}}\right)\right] \tag{2.17}$$

(3) $x \in \langle h, \infty)$,

$$T_{S3}(x,t) = \frac{2q}{k_+}\left(1 - \frac{k_-}{k_+}\right)\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n} \times \theta(t)\sqrt{t}\,\mathrm{i}\Phi_c\left(\frac{x - h + (2n+1)h\sqrt{a_1/a_2}}{2\sqrt{a_1 t}}\right) \tag{2.18}$$

where

$$t \in (-\infty, +\infty)$$
$$k_+ = \sqrt{\lambda_1\rho_1 c_1} + \sqrt{\lambda_2\rho_2 c_2}, \qquad k_- = \sqrt{\lambda_1\rho_1 c_1} - \sqrt{\lambda_2\rho_2 c_2} \tag{2.19}$$

$$a_1 = \frac{\lambda_1}{\rho_1 c_1}, \qquad a_2 = \frac{\lambda_2}{\rho_2 c_2} \tag{2.20}$$

$\mathrm{i}\Phi_c(u)$ denotes the first integral of the complementary error function $\Phi_c(u) = \mathrm{Erfc}(u)$, namely

$$\mathrm{i}\Phi_c(u) = \int_u^\infty \mathrm{Erfc}(v)\,dv$$

$$\mathrm{Erfc}(v) = \frac{2}{\sqrt{\pi}}\int_v^\infty \exp(-\varsigma^2)\,d\varsigma \tag{2.21}$$

$$\mathrm{i}\Phi_c(u) = \pi^{-1/2}\exp(-u^2) - u\,\mathrm{Erfc}(u) \tag{2.22}$$

All infinite series stated above converge very well just for several elements [1].

(b) A planar heat source producing the heat flow of a constant heat current q at an interval of time $(t - t_0, t)$ is expressed as

$$q(t) = q\theta(t_0 - t) = q\big[\theta(t) - \theta(t - t_0)\big]$$
$$q = const, \quad t_0 > 0, \quad t > 0 \tag{2.23}$$

The Laplace transformation of this function is

$$p(s) = \frac{q}{s}\big[1 - \exp(-st_0)\big] \tag{2.24}$$

Additional factor $\exp(-st_0)$ shifts the time t to $t - t_0$. The temperature fields in parts 1–3 created by the thermal source (2.23) in upper described specimen setup are determined as follows

$$T_j(x,t) = T_{Sj}(x,t) - T_{Sj}(x, t - t_0), \quad j = 1,2,3 \tag{2.25}$$

The temperature distributions settled above showed to be useful in measurements of thermal parameters by the pulse transient method [1] as well as by a step-wise method [2].

(c) The derivation of the temperature with respect to the time in case (a) gives us the temperature field for the Dirac heat source $q(t) = q\delta(t)$ located at $x = 0$. (This is due to the relation $q d\theta(t)/dt = q\delta(t)$.)

In principle this temperature fields can be used in measurements of thermal diffusivity by the Laser Pulse Method [3,4].

(d) Let us consider a planar source generating a heat flux

$$q(t) = q - q_0 \exp(-\gamma t), \quad \text{where}$$
$$q \geqslant q_0 \geqslant 0,\ \gamma > 0 \text{ are constants and } t \geqslant 0 \tag{2.26}$$

In this case we suddenly introduce a heat power $P(t)$ on a plane at $x = 0$ so that the flux $q(t)$ does not build up immediately to its final value $q = q(t \to \infty)$ but approaches it in an exponential fashion. The rate at which the flux builds up is determined by the time constant $\tau = \gamma^{-1}$.

Then the corresponding temperature fields consist of $T_{Sj}(x,t)$ given by (2.16)–(2.18) and those $T_{\gamma j}(x,t)$ produced by a planar source $-q_0 \exp(-\gamma t)$ (Appendix B) where

$$\begin{aligned} T_{\gamma 1} = \frac{q_0 \exp(-\gamma t)}{k_+\sqrt{\gamma}} \sum_{n=0}^{\infty} \left(\frac{k_-}{k_+}\right)^{2n} \biggl\{ &\mathrm{Im}\Bigl[\exp\bigl(i b_1(-x,n)\sqrt{\gamma}\bigr) \\ &\times \Phi_c\Bigl(\frac{b_1(-x,n)}{2\sqrt{t}} + i\sqrt{\gamma t}\Bigr)\Bigr] \\ &- \frac{k_-}{k_+} \mathrm{Im}\Bigl[\exp\bigl(i b_1(-x,n+1)\sqrt{\gamma}\bigr) \\ &\times \Phi_c\Bigl(\frac{b_1(-x,n+1)}{2\sqrt{t}} + i\sqrt{\gamma t}\Bigr)\Bigr]\biggr\} \end{aligned} \tag{2.27}$$

$$b_1(-x,n) = \frac{-x + 2nh\sqrt{a_1/a_2}}{\sqrt{a_1}}$$

$$\begin{aligned} T_{\gamma 2} = \frac{q_0 \exp(-\gamma t)}{k_+\sqrt{\gamma}} \sum_{n=0}^{\infty} \left(\frac{k_-}{k_+}\right)^{2n} \biggl\{ &\mathrm{Im}\Bigl[\exp\Bigl(i(2nh + x)\sqrt{\frac{\gamma}{a_2}}\Bigr) \\ &\times \Phi_c\Bigl(\frac{2nh + x}{2\sqrt{a_2 t}} + i\sqrt{\gamma t}\Bigr)\Bigr] \\ &- \frac{k_-}{k_+} \mathrm{Im}\Bigl[\exp\Bigl(i\bigl(2(n+1)h - x\bigr)\sqrt{\frac{\gamma}{a_2}}\Bigr) \\ &\times \Phi_c\Bigl(\frac{2(n+1)h - x}{2\sqrt{a_2 t}} + i\sqrt{\gamma t}\Bigr)\Bigr]\biggr\} \end{aligned} \tag{2.28}$$

$$\begin{aligned} T_{\gamma 3}(x,t) = \frac{q_0 \exp(-\gamma t)}{k_+\sqrt{\gamma}} \Bigl(1 - \frac{k_-}{k_+}\Bigr) \sum_{n=0}^{\infty} \left(\frac{k_-}{k_+}\right)^{2n} & \\ \times \mathrm{Im}\Bigl[\exp\Bigl(i\bigl(x - h + (2n+1)h\sqrt{a_1/a_2}\bigr)\sqrt{\frac{\gamma}{a_1}}\Bigr) & \\ \times \Phi_c\Bigl(\frac{x - h + (2n+1)h\sqrt{a_1/a_2}}{2\sqrt{a_1 t}} + i\sqrt{\gamma t}\Bigr)\Bigr] & \end{aligned} \tag{2.29}$$

Resulting temperature fields are

$$T_{S\gamma j}(x,t) = T_{Sj}(x,t) + T_{\gamma j}(x,t), \quad j = 1,2,3 \tag{2.30}$$

We can see that the temperature in each part of the specimen depends on the heat flux parameters q, t_0, q_0, γ, the diffusivity a_j $(= \lambda_j/\rho_j c_j)$ as well as the product of the thermo-physical parameters $\lambda_j c_j \rho_j$.

The solution for a temperature field due to heat produced by a planar Dirac source (c) or in the form of a step-wise function (a) one can also see in article [1]. Our results are different in form from those presented in [1]. But we show that they are mutual equivalent (see Appendix C). When three specimen parts are made from the same material $k_- = 0$, $k_+ \neq 0$. In that case each of infinite sums is reduced to one leading term.

3. Discussion and results

If we measure the time dependence of the temperature at a given point (say $x = h$), this in principle permits us to find the values of some thermo-physical parameters mentioned above for a selected range of temperatures. In this sense knowledge of the exact analytical formulae describing the temperature field is fundamental for many experimental methods investigating the temperature properties of materials. Moreover, the results (2.16)–(2.18) contain not only the results introduced in [1] (or some results introduced in monograph [5]) as special cases but also show their mutually intelligible connection. Above method of determining the temperature field distribution can be generalized to treat similar problems defined by different heating methods as it is shown in (d) and by modifying the boundary condition described in Eq. (2.8), assuming the form other boundary conditions remains unchanged (as it is in presence of a Dirac source (c)). Knowing the theoretical expression of temperature field distribution in a solid, one can evaluate the experimental data to extract the key thermal properties of the material. A question arises how to extract a thermal parameter (e.g., a_2) from infinite series? A procedure solving this problem is sketched in [1] for the thermal source (b).

4. Conclusions

The mathematical model considering real experimental setup is complex and its analytical solution is unknown. Therefore, some simplifying assumptions have to be postulated. A simplifying 1D-model was suggested which among others ignores the contact resistance, since it may disrupt the boundary conditions. Also exchange of heat with surroundings was neglected. These assumptions and criteria for their fulfillment in real experiment as well as a suitable time interval for evaluation of the thermophysical parameters a, c are considered and discussed in [1]. Working equations (like (16) in [1]) has been tested through experiments done by Vretenár in Ref. [1]. It is stated there that a comparison between the values a, c, λ measured by Pulse transient method in sandwich-like specimen setup and standard methods shows an excellent agreement for all parameters. These results support 1D model with adiabatic boundary condition. It seems to us that it is because of: a small heat pulse which causes small changes of temperature in a solid and short time of measurement applied in PTM.

Nevertheless, additional experimental investigation and mathematical analysis of thermal contact's influences between each part of specimen setup as well as the exchange of heat with the surroundings should be included in an improved mathematical model.

Appendix A

Let us consider a planar heat source producing the heat flow of a heat current $q(t)$in an interval of time $t > 0$. Its Laplace transformation (2.11) was denoted as $L[q(t)] = p(s)$.

The six constants A_1, B_1, A_2, B_2, A_3, B_3 obeying the transformed initial and boundary conditions (2.5a)–(2.9a) under the assumption (2.13) are:

$$A_1 = \frac{p(s)}{k_+\sqrt{s} f}\left(1 - \frac{k_-}{k_+}\exp\left(-2h\sqrt{s/a_2}\right)\right), \quad B_1 = 0 \tag{A.1}$$

$$A_2 = -\frac{p(s)}{k_+\sqrt{s} f}\frac{k_-}{k_+}\exp\left(-2h\sqrt{s/a_2}\right), \quad B_2 = \frac{p(s)}{k_+\sqrt{s} f} \tag{A.2}$$

$$A_3 = 0, \quad B_3 = \frac{p(s)}{k_+\sqrt{s} f}\left(1 - \frac{k_-}{k_+}\right)\exp\left(\sqrt{s/a_1} - \sqrt{s/a_2}\right)h \tag{A.3}$$

Here

$$f = 1 - \frac{k_-^2}{k_+^2}\exp\left(-2h\sqrt{s/a_2}\right) \tag{A.4}$$

and k_+, k_-, a_1, a_2 were determined by (2.19), (2.20).

In the transformed space, the temperature distributions are

$$\vartheta_1(x, s) = A_1(s)\exp\left(\sqrt{s/a_1}x\right) \tag{A.5}$$

$$\vartheta_2(x, s) = A_2(s)\exp\left(\sqrt{s/a_2}x\right) + B_2(s)\exp\left(-\sqrt{s/a_2}x\right) \tag{A.6}$$

$$\vartheta_3(x, s) = B_3(s)\exp\left(-\sqrt{s/a_1}x\right). \tag{A.7}$$

Appendix B

In order to find the originals $T_j(x, t)$, $j = 1, 2, 3$, to the pictures (A.5)–(A.7), we must first expand the function f^{-1} into the infinite series

$$f^{-1} = \left\{1 - \frac{k_-^2}{k_+^2}\exp\left(-2h\sqrt{s/a_2}\right)\right\}^{-1} = \sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\exp\left(-2nh\sqrt{s/a_2}\right) \tag{B.1}$$

valid for

$$\frac{k_-^2}{k_+^2}\exp\left(-2h\sqrt{s/a_2}\right) < 1 \quad \text{(by definition } |k_-/k_+| \leqslant 1) \tag{B.2}$$

and select a method of heating. This is determined by the function $q(t)$. In a simple case when $q(t) = const = q$ for the time $t > 0$ its Laplace transformation is $p(s) = q/s$.

Substituting for $p(s)$ and (B.1) in formula (A.1) and subsequently in (A.5) we find

$$\vartheta_1 = \frac{q}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\frac{1}{s\sqrt{s}}\Big[\exp\left(-\left(2n\sqrt{s/a_2}h - \sqrt{s/a_1}x\right)\right) - \frac{k_-}{k_+}\exp\left(-\left(2(n+1)\sqrt{s/a_2}h - \sqrt{s/a_1}x\right)\right)\Big]. \tag{B.3}$$

One possible way for determining $T_{S1}(x, t)$ is to use the tables of the inverse Laplace transformation (e.g., [5] p. 494, Appendix V, formula 9)

$$\begin{aligned} L^{-1}\left[\frac{\exp(-b_1\sqrt{s})}{s\sqrt{s}}\right] &= 2\sqrt{t}\,\mathrm{i}\Phi_c\left(\frac{b_1(n,-x)}{2\sqrt{t}}\right) \\ &= 2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{b_1^2(n,-x)}{4t}\right) \\ &\quad - b_1(n,-x)\Phi_c\left(\frac{b_1(n,-x)}{2\sqrt{t}}\right) \end{aligned} \tag{B.4}$$

where

$$b_1(n, -x) = 2nh/\sqrt{a_2} - x/\sqrt{a_1} = \frac{-x + 2nh\sqrt{a_1/a_2}}{\sqrt{a_1}} \geqslant 0 \tag{B.5}$$

Thus the temperature field in part 1 of the specimen setup is then given as

$$T_{S1}(x, t) = \frac{2q}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\sqrt{t}\Big[\mathrm{i}\Phi_c\left(\frac{b_1(n,-x)}{2\sqrt{t}}\right) - \frac{k_-}{k_+}\mathrm{i}\Phi_c\left(\frac{b_1(n+1,-x)}{2\sqrt{t}}\right)\Big]$$
$$x \in (-\infty, 0\rangle,\ t > 0$$

which is identical with (2.16). In this way one can also find originals $T_{S2}(x, t)$, $T_{S3}(x, t)$ given by the formulae (2.17), and (2.18).

(b) If we have a thermal source of a constant power pulse with duration $t_0 > 0$ then $q(t) = q\theta(t_0 - t) = q[\theta(t) - \theta(t - t_0)]$, $t_0 > 0$, $t > 0$ and the Laplace transformation of this pulse is $p(s) = q(1 - \exp(-st_0))/s$. Additional factor $\exp(-st_0)$ shifts the time t to $t - t_0$ because of

$$L^{-1}\left[\exp(-st_0)\frac{\exp(-b_1\sqrt{s})}{s\sqrt{s}}\right] = 2\sqrt{t - t_0}\,\mathrm{i}\Phi_c\left(\frac{b_1(n,-x)}{2\sqrt{t - t_0}}\right), \quad t > 0,\ t > t_0 \tag{B.6}$$

$$T_1(x, t) = T_{S1}(x, t) - T_{S1}(x, t - t_0)$$

This expression determines the temperature field in part 1 created by the thermal source in the case (b) in upper described specimen setup.

By analogy $T_j(x, t) = T_{Sj}(x, t) - T_{Sj}(x, t - t_0)$, $j = 2, 3$, determines the temperature field in part 2 and 3, respectively.

(d) If we consider a source $q(t) = q - q_0\exp(-\gamma t)$ where $q \geqslant q_0 \geqslant 0$ and $\gamma > 0$ are constant, the Laplace transformation of this function is

$$p(s) = \frac{q}{s} - \frac{q_0}{s + \gamma} \tag{B.7}$$

The term q/s has been already discussed. Thus the problem is reduced to finding the temperature field corresponding to the Laplace transformation temperature of the form

$$L[f(t)] = \frac{\exp(-b\sqrt{s})}{\sqrt{s}(s+\gamma)}, \quad b \text{ is real and positive} \tag{B.8}$$

Its inverse Laplace transformation is ([5], Appendix V)

$$\begin{aligned} f(t) &= L^{-1}\left[\frac{\exp(-b\sqrt{s})}{\sqrt{s}(s+\gamma)}\right] \\ &= -\frac{\exp(-\gamma t)}{\sqrt{\gamma}} \operatorname{Im}\left[\exp(ib\sqrt{\gamma})\Phi_c\left(\frac{b}{2\sqrt{t}} + i\sqrt{\gamma t}\right)\right], \\ &\gamma > 0 \end{aligned} \tag{B.9}$$

where $i = \sqrt{-1}$ (imaginary unit) and $\Phi_c = \operatorname{Erfc}(z)$ is the complementary error function of complex variable

$$z = \frac{b}{2\sqrt{t}} + i\sqrt{\gamma t} \tag{B.10}$$

Factors b are the same for parts 1–3 as those appearing in (2.16)–(2.18). Then, the temperature field generated by a thermal source $-q_0\exp(-\gamma t)$ located at $x = 0$ in each part of the specimen setup is described by formulae (2.27)–(2.29).

Appendix C

We show that our results giving the temperature fields in cases (a)–(c) are equivalent with Vretenár's [1]. The temperature field created in the middle part 2 of a specimen by a planar thermal source (a) ($q = const$) is given by (2.17)

$$\begin{aligned} T_{S2}(x,t) = \frac{2q\sqrt{t}}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\bigg[\mathrm{i}\Phi_c\left(\frac{2nh+x}{2\sqrt{a_2t}}\right) \\ - \frac{k_-}{k_+}\mathrm{i}\Phi_c\left(\frac{2(n+1)h-x}{2\sqrt{a_2t}}\right)\bigg] \end{aligned}$$

The same field is given in Ref. [1] by formula (16) as follows

$$\begin{aligned} T_{s2}(x,t) = \frac{2q\sqrt{t}}{k_+}\bigg[\mathrm{i}\Phi_c\left(\frac{x}{2\sqrt{a_2t}}\right) \\ + \frac{k_-^2}{k_+^2}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\mathrm{i}\Phi_c\left(\frac{2(n+1)h+x}{2\sqrt{a_2t}}\right) \\ - \frac{k_-}{k_+}\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\mathrm{i}\Phi_c\left(\frac{2(n+1)h-x}{2\sqrt{a_2t}}\right)\bigg] \end{aligned}$$

(16) in Ref. [1]

(assuming that $k_1 \to k_-$, $k_2 \to k_+$). We are going to show mutual equivalence both formulae.

The sum appearing in (2.17) can be written as

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\mathrm{i}\Phi_c\left(\frac{2nh+x}{2\sqrt{a_2t}}\right) \\ &\quad = \mathrm{i}\Phi_c\left(\frac{x}{2\sqrt{a_2t}}\right) + \sum_{r=1}^{\infty}\left(\frac{k_-}{k_+}\right)^{2r}\mathrm{i}\Phi_c\left(\frac{2rh+x}{2\sqrt{a_2t}}\right) \end{aligned} \tag{C.1}$$

Then it holds

$$\begin{aligned} &\sum_{r=1}^{\infty}\left(\frac{k_-}{k_+}\right)^{2r}\mathrm{i}\Phi_c\left(\frac{2rh+x}{2\sqrt{a_2t}}\right) \\ &\quad = \left(\frac{k_-}{k_+}\right)^2\sum_{r=1}^{\infty}\left(\frac{k_-}{k_+}\right)^{2r-2}\mathrm{i}\Phi_c\left(\frac{2rh+x}{2\sqrt{a_2t}}\right) \end{aligned}$$

and substitution $n = r-1, r = 1,2,3,\ldots \Rightarrow n = 0,1,2,\ldots$ permits to write that sum as

$$= \left(\frac{k_-}{k_+}\right)^2\sum_{n=0}^{\infty}\left(\frac{k_-}{k_+}\right)^{2n}\mathrm{i}\Phi_c\left(\frac{2(n+1)h+x}{2\sqrt{a_2t}}\right) \tag{C.2}$$

Now, we can rewrite (2.17) into the form given by the formula (16) in Ref. [1].

In that manner (withdrawing the term $n = 0$ which does not contain factor k_-/k_+ from the infinite sum and multiplying rest of the sum by $(k_-/k_+)^2(k_-/k_+)^{-2} = 1$) one can show equivalence here presented formulae and corresponding formulae introduced in [1].

References

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